**Problem 10D,4**
Suppose $T$ is a bounded operator on a Hilbert space $V$ and $U$ is a closed subspace of $V$. Prove that the following are equavalent:

- $U, U^\perp$ are invariant subsapces for $T$.
- $U, U^\perp$ are invariant subspaces for $T^*$
- $TP_U = P_U T$.

*Proof.* a $\Longleftrightarrow$ b: Let $a \in U$, we know that for any $b \in U6\perp, < T^*a, b >=< a, Tb >= 0$ thus $T^*a \in U$ which means that $U$ is $T^*$ invariant. Similar assertion holds for $U^\perp$.

a $\longrightarrow$ c: For any $a \in V$, we can assume its decompostion is $a = b + c, b \in U, c \in U^\perp$. Then

$$TP_U(a) = T(b) = P_U T(b) = P_U T(b + c) = P_u T(a).$$

c $\longrightarrow$ a: For any $a \in U, b \in U^\perp$,

$$< Ta, b >=< TP_U a, b >=< P_U Ta, b >= 0,$$

which implies $Ta \in U$. Other assertion follows similarly. $\square$

**Problem 10D,7**
Suppose $T$ is a self-adjoint compact operator on a Hilbert space but only has finitely many distinct eigenvalues. Prove that $T$ has finite dimenisonal range.

*Proof.* This follows directly from 10.106. $\square$

**Problem 10D,12**
Suppose $T$ is a compact operator on a Hilbert space . Prove that $s_1(T) = \|T\|$.

*Proof.* Recall that

$$\|T\| = \sup_{\|x\|=1} \|Tx\|.$$

and also 10.113. We know that $\|T^*T\| = \|T\|^2$.For any $x \in V$ with norm 1, decompose

$$x = x_0 + < x, e_1 > e_1 + < x, e_2 > e_2 + \dots$$

as in 10.113, where $e_0$ is in null of $T$. Then

$$\|T^*Tx\|^2 = \sum_{k=1}^{\infty} | < x, e_k > s_k^2|^2 \leq s_1^4 \sum_{k=1}^{\infty} | < x, e_k > |^2 \leq s_1^4$$

with equality holds when $x = e_1$. This finishes the proof. $\square$

**Problem 10D,17**
Suppose $T$ is a compact operator on a Hilbert space with singular value decomposition

$$Tf = \sum_{k\in\Omega} s_k < f, e_k > h_k,$$

for all $f \in V$. Prove that

$$T^*f = \sum_{k\in\Omega} s_k < f, h_k > e_k$$

for all $f \in V$.

*Proof.* Following the proof in 10.113, we know that

$$T^* h_k = \frac{T^* T e_k}{s_k} = \frac{s_k^2 e_k}{s_k} = s_k e_k.$$

And if $v \in V$ such that $< v, h_k >= 0$ for any $k$, then $< T^* v, e_k >= 0$ which implies $T^* v = 0$.So

$$T^* f = T^*(\sum_{k \in \Omega} < f, h_k > h_k) = \sum_{k \in \Omega} < f, h_k > T^* h_k = \sum_{k \in \Omega} s_k < f, h_k > e_k$$

□

**Problem 11A,4**
Suppose $f \in L^1(\partial D), z \in \partial D$ and f is continuous at $z$. Prove that

$$\lim_{r \uparrow 1} (P_r f)(z) = f(z)$$

*Proof.* Following the proof in 11.18, we find we only need to show that

$$\lim_{r \uparrow 1} \int_{\{\zeta \in \partial D : |\zeta - z| \geq \delta\}} |f(\zeta) - f(z)| P_r(z\bar{\zeta}) = 0$$

This is easy to see since on the integral domain, $P_r(z\bar{\zeta})$ uniformly converges to 0 so we can use dominated convergence theorem to finish the proof. □

**Problem 11A,6**
Prove that for each $p \in [1, \infty)$, there exists $f \in L^1(\partial D)$ such that

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^p = \infty$$

*Proof.* Theorem 3.3.4 in Loukas Grafakous' book "Classical Fourier Analysis" is a stronger result. Interested people can find the construction there. □

**Problem 11A,10**
Suppose $f : \partial D \to \mathbb{C}$ is three times continuously differentiable. Prove that for all $z \in \partial D$,

$$f^{[1]}(z) = i \sum_{n=-\infty}^{\infty} n \hat{f}(n) z^n.$$

*Proof.* Follow the proof in 11.27, three time differentiable implies for all $z \in \partial D$,

$$\sum_{n=-\infty}^{\infty} |n \hat{f}(n) z^n| < \infty$$

Then apply the Possion summation formula to $f^{[1]}$ and recall 11.26 $(f^1)\hat{}(n) = in\hat{f}(n)$, this finishes the proof. (The proof is word by word as in the proof of 11.27.) □

**Problem 11A,12**
Define $f : \partial D \to \mathbb{R}$ by

$$f(z) = \begin{cases} 1 & Imz > 0 \\ 0 & Imz = 0 \\ -1 & Imz < 0 \end{cases}$$

- Show that if $n \in \mathbb{Z}$, then

$$\hat{f}(n) = \begin{cases} \frac{-2i}{n\pi} & n\ odd \\ 0 & n\ even \end{cases}$$

- Show that for $r \in [0, 1), z \in \partial D$,

$$(P_r f)(z) = \frac{2}{\pi} \arctan \frac{2rImz}{1 - r^2}$$

- Verify $\lim_{r \uparrow 1}(P_r f)(z) = f(z)$ for every $z \in \partial D$.
- Prove that $P_r f$ does not converge uniformly to $f$ on $\partial D$.

*Proof.*

- When $n$ is even, by symmetry, the integral is zero. when $n$ is odd,

$$\hat{f}(n) = \int_0^\pi \frac{\cos nt - i \sin nt}{2\pi} dt - \int_{-\pi}^0 \frac{\cos nt - i \sin nt}{2\pi} dt = \frac{-2i}{n\pi}$$

- From 11.11, we know that

$$(P_r f)(z) = \sum_{n\ odd} r^{|n|} \frac{-2i}{n\pi} z^n = \frac{-2i}{\pi} \sum_{k=0}^{\infty} \frac{(rz)^{2k+1} - (r\bar{z})^{2k+1}}{2k+1} = \frac{2}{\pi} \arctan \frac{2rImz}{1 - r^2}$$

- This is obvious.
- Let $n$ be any positive integer. For $1 - r = \frac{1}{n}$, set $z$ such that $Imz > 0$, $\frac{2rImz}{1-r^2} = 1$, then $|1 - f(z)| = \frac{1}{2}$. This shows that the convergence is not uniformly.
 ☐